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Periodic orbit analysis of billiard level dynamics

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Experimentally determined eigenvalue velocities in a Sinai billiard were used to establish a relation between the Pechukas-Yukawa level dynamics and periodic orbit theory. The first one explains easily the observed Maxwellian velocity distributions, whereas the second one accounts for velocity correlations between distant levels, and a conspicuous increase of the averaged velocities with energy.

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In investigations of the quantum-mechanical spectra of classically chaotic systems one can identify three main approaches: random matrix theory, the Coulomb gas analogy of level motion, and periodic orbit theory. The first concentrates on statistical properties of the fluctuations of eigenvalues around some mean and predicts, in analogy to observations in nuclear physics, universal behavior depending only on whether the Hamiltonian is real symmetric, complex Hermitian or symplectic [1]. The second approach studies the motion of eigenvalues under parameter change [2,3]. Equations of motion can be derived reminiscent of those for a one-dimensional Coulomb gas, once the eigenvalues are identified with the positions of the particles. "Random" behavior can now be studied using methods from statistical physics. The third approach makes use of the relation between quantum spectra and classical periodic orbits as embodied in semiclassical trace formulas [4,5]. There is some hope that it can provide the extension of WKB quantization to chaotic systems, but perhaps more importantly it predicts correlations in eigenvalues due to classical periodic orbits [6].

The relation of the latter two theories to random matrix theory has been clarified to a considerable extent in Refs. [1–3,7]. Here we focus on the relation between periodic orbit theory and level dynamics. The predictions of the theory are then tested experimentally using eigenfrequency spectra of billiard-shaped microwave resonators [8,9] of varying length. In particular, we will use periodic orbit theory to explain an unexpected linear increase of eigenvalue velocities with energy.

We begin with a brief summary of the Pechukas-Yukawa model [2,3] of level dynamics. The original theory assumes a Hamiltonian with a linear parameter dependence and is therefore not directly applicable to billiards, since here the Hamiltonian $\tilde{H}=-\hbar^2/(2m)\Delta$ is fixed, and possible parameter dependences enter only through the change of the shape. It is always possible, however, to transform a simply connected region to a standard shape, e.g., the unit circle, by means of a conformal mapping [10]. By this the parameter dependence is shifted from the boundary conditions to the Hamiltonian,

$$H(\lambda)=g(\lambda)\tilde{H} \ , \qquad (1)$$

where $g(\lambda)$ is the functional determinant of the mapping. For sufficiently small changes of the parameter in some interval $\lambda\in[\lambda_0,\lambda_0+\Delta\lambda]$ one can expand to first order in λ, arriving at

$$H(\lambda)=H_0+(\lambda-\lambda_0)V \ , \qquad (2)$$

where $H_0=H(\lambda_0)$ and $V=\partial H/\partial\lambda|_{\lambda=\lambda_0}$. Let $x_n(\lambda)$ be the eigenvalues, $|n(\lambda)\rangle$ the associated eigenvectors, $V_{nm}=\langle n(\lambda)|V|m(\lambda)\rangle$ the matrix elements, and

$f_{nm}=|x_n-x_m|V_{nm}$. One can derive equations of motion for these quantities [2,3], of which we will cite for later reference the one for the velocities only,

$$\dot{x}_n=\frac{\partial x_n}{\partial\lambda}=V_{nn} . \tag{3}$$

The equations of motion have an infinite number of conservation laws, the simplest of which are

$$E=\tfrac{1}{2}\sum_n \dot{x}_n^2+\tfrac{1}{2}\sum_{n\neq m}\frac{|f_{nm}|^2}{(x_n-x_m)^2} ,$$
$$Q=\tfrac{1}{2}\sum_{n\neq m}|f_{nm}|^2 . \tag{4}$$

The first one is reminiscent of the total energy of a one-dimensional gas of particles at positions x_n with velocities $\dot{x}_n$, interacting via a pairwise repulsive (distance)$^{-2}$ potential, the strength being given by $|f_{nm}|^2$. Within statistical mechanics one would describe a subset of levels by a canonical distribution in phase space,

$$\rho(x_n,\dot{x}_n,f_{nm})=\frac{1}{Z}\exp(-\mu E-\sigma Q) , \tag{5}$$

with Lagrange parameters μ and σ, normalized by the canonical partition sum Z.

It should be noted that E and Q are no longer constants of motion if higher-order terms in λ are included in Eq. (2). Observing that E can be written as the trace over $\frac{1}{2}V^2$, one finds

$$E(\lambda)=\tfrac{1}{2}\mathrm{tr}\left[\frac{\partial H}{\partial\lambda}\right]^2 +(\lambda-\lambda_0)\mathrm{tr}\left[\frac{\partial H}{\partial\lambda}\frac{\partial^2 H}{\partial\lambda^2}\right]+\cdots , \tag{6}$$

where all derivatives are taken at $\lambda=\lambda_0$ (a similar calculation can be performed for Q). If, however, already a small variation in λ leads to a new arrangement of the levels (in the experiment a relative change in the billiard length of 10^{-2} was sufficient to produce completely new spectra, see Fig. 1) then one can expect that the phase-space density relaxes quickly to the canonical one given by Eq. (5). A similar reasoning can probably be applied to other Hamiltonians nonlinear in λ (as studied, e.g., in Refs. [11,12]).

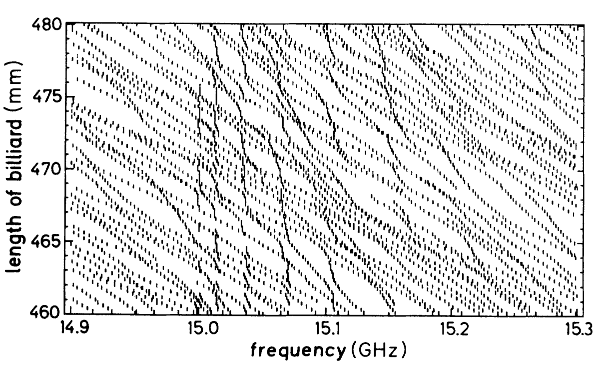

FIG. 1. Part of the microwave spectrum of a quartered Sinai billiard as a function of length. The overall decrease of the eigenfrequencies with increasing length reflects the increase of the density of states with area. The peculiar solitonlike structures [15] in the middle appear every 0.75 GHz and are associated with the dominating bouncing ball orbit labeled 1 in Fig. 3 (compare also Fig. 4 of Ref. [8]).

Gutzwiller's periodic orbit theory in its basic formulation relates modulations in the density of states $\rho(k)=\sum_n\delta(k-k_n)$ to periodic orbits [4]. For billiards, one finds for the density of states

$$\rho(k)=\rho_0(k)+\sum_p L_p w_p \exp(iL_p k) , \tag{7}$$

where k is the wave number ($k=\sqrt{E}$, assuming $\hbar^2/2m=1$), p labels periodic orbits including multiple traversals, L_p is the total length of the orbit, w_p depends on the Maslov index and the stability of the orbit. The smooth term is given by the Weyl formula,

$$\rho_0(k)=\frac{A}{2\pi}k-\frac{L}{4\pi} , \tag{8}$$

with A the area of the billiard and L its circumference [13]. The advantage of considering the density as a function of wave number rather than energy is that the periodic orbit content of the eigenvalue distribution can be obtained by a simple Fourier transform.

To connect periodic orbit theory to level dynamics, note that the velocity of an eigenvalue x_n is given by the diagonal matrix element V_{nn} [see Eq. (3)]. The velocities of eigenvalues x_n are related to those of the wave numbers k_n by $V_{nn}=\dot{x}_n=2k_n\dot{k}_n$. The density of eigenvalue velocities is given by

$$\rho_v(k)=\sum_n V_{nn}\delta(k-k_n) . \tag{9}$$

To obtain a periodic orbit expression for $\rho_v(k)$ one could apply the results of Ref. [5] for expressions of the form $\sum_n\langle n|A|n\rangle\delta(k-k_n)$, if only the operator corresponding to the velocities was known. Here, we can proceed differently, starting from the integrated density of states,

$$N(k)=\sum_n\theta(k-k_n) .$$

Its derivative with respect to the parameter λ in the Hamiltonian gives, up to a factor, the density of eigenvalues velocities,

$$\frac{\partial N}{\partial\lambda}(k)=-\sum_n\frac{\partial k_n}{\partial\lambda}\delta(k-k_n)=-\frac{1}{2k}\rho_v(k) .$$

A periodic orbit expression for the integrated density of states can be obtained by integration of Eq. (7). Assuming that the variations of the phase factors with λ are much larger than the variations of the amplitudes, one gets

$$\rho_v(k)=-2k^2\sum_p \dot{L}_p w_p \exp(iL_p k) , \tag{10}$$

which shows that the density of velocities should increase linearly with energy $E=k^2$.

To test this, we have performed experiments on a resonator shaped as a quartered, i.e., desymmetrized, Sinai

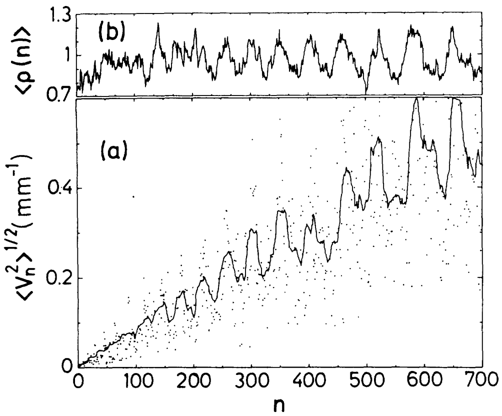

FIG. 2. (a) Absolute values $|v_n|$ of the eigenvalue velocities for a quartered Sinai billiard (points) and $\langle v_n^2 \rangle^{1/2}$ averaged over 20 neighbored eigenvalues (continuous line). (b) Experimental density of states $\langle \rho(x) \rangle$ averaged over 20 neighboring eigenvalues. The spectra are normalized to a constant density of 1.

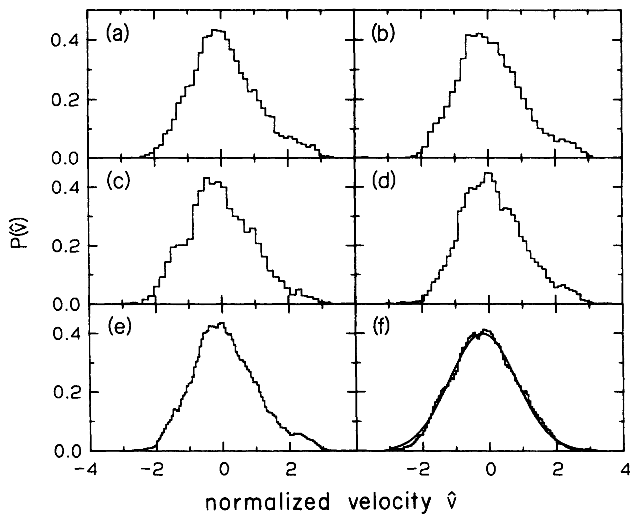

FIG. 4. Distribution of normalized velocities $\hat{v}_n = v_n / \langle v_n^2 \rangle^{1/2}$ in the frequency ranges (a) 0.5–8.0 GHz, (b) 8.0–10.5 GHz, (c) 10.5–12.5 GHz, (d) 12.5–15.0 GHz, (e) 0.5–15.0 GHz, (f) 0.5–15.0 GHz. In (a)–(e) all eigenfrequencies within the respective ranges were considered; in (f) only eigenfrequencies not influenced by the bouncing ball were taken. The solid line is a Gaussian.

billiard with dimensions $a = 460–480$ mm, $b = 200$ mm, $r = 70$ mm. The height of the resonator was $d = 8$ mm. For frequencies $\nu < c/(2d) = 18.7$ GHz the billiard can be considered as two-dimensional and the time-independent Schrödinger equation and electromagnetic wave equation for the electric-field strength are completely equivalent [8]. The length of the billiard was varied in steps between 0.2 and 1 mm, and some 20 spectra each containing about 700 eigenvalues were obtained in a very short time. Figure 1 shows an example. Parts of the spectrum are lost for two reasons: (i) some eigenvalues disappear temporarily because of the passing of a nodal line through the position of the coupling wire [9], and (ii) eigenfrequencies can no longer be separated if their distance is smaller than the linewidth of several MHz. Since all eigenvalues were registered as a function of length, all missing eigenvalues could be recovered by interpolation, thus reducing the loss to zero within the limits of error.

Because of the Weyl formula (8) the mean density of states $\rho_0(k)$ increases in the leading term linearly with k and is proportional to the billiard area. This leads to the overall decrease of eigenfrequencies with length. For further discussion the spectra are unfolded to a constant mean density of states using Eq. (8). Figure 2(a) shows the absolute value of the velocities $v_n = \dot{x}_n$ of the eigenvalues as well as $\langle v_n^2 \rangle^{1/2}$, averaged over 20 eigenvalues. One observes periodic modulations superimposed on a linear increase. The oscillations of $\langle v_n^2 \rangle^{1/2}$ are correlated with corresponding oscillations in the density of states, as can be seen in Fig. 2(b). The periodic modulations reflect the presence of periodic orbits as was discussed in Ref. [14].

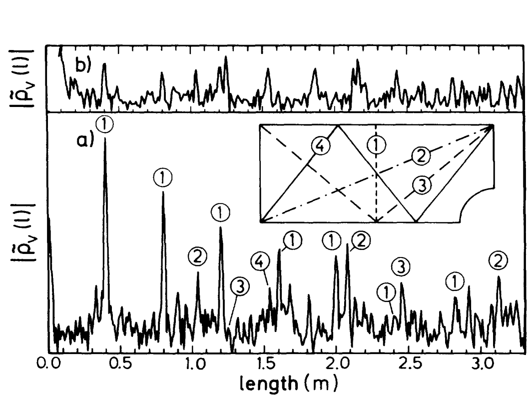

FIG. 3. (a) Absolute value of the Fourier transform $\tilde{\rho}_v(l) = \int (2k^2)^{-1} \rho_v(k) \exp(-ikl) dk$ of the eigenvalue velocities ($a = 478$ mm, $b = 200$ mm, $r = 70$ mm). (b) Absolute value of the Fourier transform $\tilde{\rho}(l) = \int \rho(k) \exp(-ikl) dk$ of the density of states. The resonances both of $\tilde{\rho}(l)$ and $\tilde{\rho}_v(l)$ can be associated with periodic orbits (see inset).

The contribution of the different orbits can be projected out of $\rho(k)$ and $(2k^2)^{-1} \rho_v(k)$ by a Fourier transformation. The result is shown in Figs. 3(a) and 3(b). The most prominent resonances belong to the bouncing ball and its higher harmonics, but contributions of several other orbits are also clearly seen. The ratio of amplitudes of correspondings peaks in Figs. 3(a) and 3(b), respectively, should be equal to $-\dot{L}_p / L_p$ [see Eqs. (7) and (10)]. We checked this qualitatively for the resonances marked by numbers in Fig. 3. In all cases the ratio of the two amplitudes showed up to be real, up to phase deviations of the order of ± 0.02 from 0 or π. The sign of the ratio indicated correctly whether the length of the orbit in question increases or decreases with billiard length. It should be noted that the unfolded spectra correspond to billiards of constant area, i.e., an increase of the length is compensated by a corresponding decrease of the width. Therefore the lengths of orbits 1 and 4 decrease with billiard length (see inset of Fig. 3), whereas the length of orbit 2 increases. The length of orbit 3 is nearly independent of the billiard length. This explains easily the nonexistence of the corresponding peak at 1.25 m in Fig. 3(a), while it is clearly seen in Fig. 3(b). The peak corresponding to the first harmonic of orbit 3, however, is present in both Fourier transforms, suggesting that for a quantitative analysis the parameter dependences of the amplitudes, neglected in the derivation of Eq. (10), have to be taken into account also.

The above findings show that eigenvalue dynamics increases with energy, even if the energies are unfolded to a constant mean spacing. The situation is different, e.g., for periodically kicked tops where all eigenphases are equivalent with respect to level dynamics [1]. This suggests the introduction of a local time $\tau=\langle x_n \rangle\lambda$. If the velocities are defined in terms of τ, the linear increase shown in Fig. 2 cancels out, and level dynamics becomes independent of energy. This fact is further corroborated by the observed velocity distributions. Figures 4(a)–4(d) show distribution histograms for different frequency ranges. The distributions are normalized to a quadratically averaged velocity of 1. All histograms look identical within the limits of error. In Fig. 4(e) the velocity distribution histogram for all eigenenergies is shown. From the Yukawa conjecture (5) a Gaussian, i.e., Maxwellian velocity distribution is expected. The histogram, however, shows a distinct shoulder at positive velocities. This behavior is easily correlated with the solitonlike structures [15] seen in Fig. 1, and their weak dependences on the billiard length. If the spectrum is unfolded to constant density, the bouncing ball eigenvalues experience positive drift velocities giving rise to the observed asymmetry. If the eigenvalues in these regions are omitted from the histogram, good agreement with the Gaussian prediction is found [Fig. 4(f)]. Analogous influences of the bouncing ball orbit on nearest-neighbor spacing distribution are known from the stadium billiard [14,16].

This paper concentrated on the relations between level dynamics and periodic orbit theory. We would like to mention that beyond it the measurements allow tests of random matrix predictions, e.g., on asymptotic curvature distributions [17] or on distributions of closest approach distances [18]. In all these cases random matrix theory could completely account for the experimental results. Details will be presented in a forthcoming paper.

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